Asymptotic behaviors of singular homogeneous solutions of some partial differential operators in the complex domain (Microlocal Analysis and Related Topics)

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§1 Introduction

Let $L(z,\partial_z)$ be a linear partial differential operator with holomorphic coefficients in a neighborhood of $z=0$ in $\mathbb{C}^{d+1}$ and K be a nonsingular complex hypersurface through the origin. The coordinate of $\mathbb{C}^{d+1}$ is denoted by $(z_0, z_1, \cdots, z_d)$ and chosen such that $K=\{z_0=0\}$. Let $u(z)$ be a solution of $L(z,\partial_z)u(z)=0$, which is not necessary holomorphic on K. The existence of singular solutions is studied by many mathematician (for example see [2],[3], [5] and [9]). The purpose of the present paper is to introduce a class of partial differential operators and to study the asymptotic behaviors as $z_0 \to 0$ of singular solutions of $L(z,\partial_z)u(z)=0$ for $L(z,\partial_z)$ belonging to this class. In general there are many singular homogeneous solutions, hence we restrict solutions by adding a condition of the growth order of its singularities to them. So we treat solutions with at most some exponential order singularities on K which is given the constant γ defined by (2.2). It is the main result that we can give the asymptotic terms of solutions as $z_0 \to 0$ and the remainder term with Gevrey type estimate. The Gevrey exponent is also determined by γ. The operators considered here have useful examples, so the main result of Ōuchi [6] follows from that in this paper and those of Mandai [4] and Tahara [10] concerning the structure of homogeneous solutions of Fuchsian operators also do in some sense. So the results here are extensions of results of [4] and [10] to non-Fuchsian operators in some sense.

The details of this paper will be appeared in Ōuchi [8].

§2 Operators and and Definitions

In this section let us introduce a class of operators studied in this paper and give some definitions. Let $L(z,\partial_z)$ be an m-th order linear partial differential operator with holomorphic coefficients in a domain in $\mathbb{C}^{d+1}$ of the form:

$$(2.1)\qquad \begin{cases} L(z,\partial_z) = A(z,\partial_{z_0}) + B(z,\partial_z), \\ A(z,\partial_{z_0}) = \sum_{i=0}^{k} a_i(z')(z_0\partial_{z_0})^i, \\ B(z,\partial_z) = \sum_{|\alpha|\le m} b_\alpha(z)\partial_z^\alpha, \quad z=(z_0,z_1,\cdots,z_d)=(z_0,z'). \end{cases}$$

Let $j_\alpha \in \mathbb{N}$ such that $b_\alpha(z) = z_0^{j_\alpha}\tilde{b}_\alpha(z)$ with $\tilde{b}_\alpha(0,z') \not\equiv 0$ on $K = \{z_0 = 0\}$ provided $b_\alpha(z) \not\equiv 0$. Let us assume in this paper that $L(z,\partial)$ satisfies the following conditions (A) and (B),

$$\text{(A)}\qquad a_k(0) \neq 0,$$

$$\text{(B)}\qquad j_\alpha - \alpha_0 > 0 \quad \textit{for all } \alpha.$$

We define an important constant γ by

$$(2.2)\qquad \gamma := \begin{cases} \min\{\dfrac{j_\alpha - \alpha_0}{|\alpha| - k};\ |\alpha| > k\} & \textit{if } k < m, \\ +\infty & \textit{if } k = m. \end{cases}$$

and a polynomial $\chi(\lambda, z')$ by

$$(2.3)\qquad \chi(\lambda,z') = \sum_{i=0}^{k} a_i(z')\lambda^i.$$

Let us give examples, which show that the class of operators considered in this paper contains useful examples.

(1). Let

$$(2.4)\qquad P(z,\partial_z) = \partial_{z_0}^k + \sum_{\substack{|\alpha|\le m \\ \alpha_0<k}} a_\alpha(z)\partial_z^\alpha \quad (m > k).$$

$P(z,\partial_z)$ is a linear partial differential operator with order m and is of the normal form with respect to ∂_{z_0}. By multiplying $P(z,\partial_z)$ by z_0^k, consider $z_0^k P(z,\partial_z)$. Then $z_0^k P(z,\partial_z)$ satisfies (A) and (B), by setting $A(z_0,\partial_{z_0}) = z_0^k\partial_{z_0}^k$ and $B(z,\partial_z) = \sum_{|\alpha|\le m,\alpha_0<k} z_0^k a_\alpha(z)\partial_z^\alpha$.

(2). Let $P(z,\partial_z)$ be an m-th operator of Fuchsian type weight $(m-h)$ in the sense of Baouendi-Goulaouic [1]. Then $z_0^{m-h}P(z,\partial_z)$ belongs to the class we consider and $\gamma = +\infty$.

(3). We give a concrete example. Let $z = (z_0, z_1) \in \mathbb{C}^2$ and

$$L(z, \partial_z) = z_0\partial_{z_0} - a(z) + z_0^j c(z)\partial_{z_1}^m, \tag{2.5}$$

where $j \geq 1$ and $c(0, z_1) \not\equiv 0$. Then $\chi(\lambda, z_1) = \lambda - a(0, z_1)$ and $\gamma = j/(m - 1)$ $(m > 1)$, $\gamma = +\infty$ $(m = 1)$.

Let us introduce function spaces on the sectorial region $U(\theta)$ for our aim.

Definition 2.1. *$\mathcal{O}_{(\kappa)}(U(\theta))$ is the set of all $u(z) \in \mathcal{O}(U(\theta))$ such that for any $\varepsilon > 0$ and any θ' with $0 < \theta' < \theta$*

$$|u(z)| \leq M \exp(\varepsilon |z_0|^{-\kappa}) \quad \text{for} \quad z \in U(\theta') \tag{2.6}$$

holds for some constant $M = M(\varepsilon, \theta')$. We put $\mathcal{O}_{(+\infty)}(U(\theta)) = \mathcal{O}(U(\theta))$.

Definition 2.2. *$\mathcal{O}_{temp,c}(U(\theta))$ is the set of all $u(z) \in \mathcal{O}(U(\theta))$ such that for any θ' with $0 < \theta' < \theta$*

$$|u(z)| \leq M|z_0|^c \quad \text{for} \quad z \in U(\theta') \tag{2.7}$$

holds for some constant $M = M(\theta')$.

Set $\mathcal{O}_{temp}(U(\theta)) = \cup_{c\in\mathbb{R}}\mathcal{O}_{temp,c}(U(\theta))$, which is the set of all holomorphic functions on $U(\theta)$ having singularities on $z_0 = 0$ with fractional order. We also say that $u(z) \in \mathcal{O}(U(\theta))$ is tempered singular on $(U(\theta)$, provided $u(z) \in \mathcal{O}_{temp}(U(\theta))$.

§3 Behaviors of singular solutions

Now let us return to the equation $L(z, \partial_z)u(z) = 0$, $u(z) \in \mathcal{O}(U(\theta))$. In order to study the behaviors of solutions more concretely we restrict the growth properties of singularities, that is, we assume $u(z) \in \mathcal{O}_{(\gamma)}(U(\theta))$ in this paper, where γ is defined by (2.2). Firstly we show that it follows from this assumption that the singularities of solutions are less irregular.

As for the zeros of $\chi(\lambda, z')$ it follows from the condition (A), that there are constants $r' > 0$, a_0, a_1 and b such that $\chi(\lambda, z') = 0$ has k roots for $z' \in V' = \{|z'| \leq r'\}$ and

$$\{\lambda;\ \chi(\lambda, z') = 0\} \subset \{\lambda;\ a_0 \leq \Re\lambda \leq a_1,\ |\Im\lambda| \leq b\}. \tag{3.1}$$

holds. Then we have, by using the constant a_0,

Theorem 3.1. ([7]). *Let $u(z) \in \mathcal{O}_{(\gamma)}(U(\theta))$ be a solution of $L(z, \partial_z)u(z) = f(z) \in \mathcal{O}_{temp,c}(U(\theta))$. Then there is a polydisk V centered at $z = 0$ such that $u(z) \in \mathcal{O}_{temp,c'}(V(\theta))$ for any $c' < \min\{c, a_0\}$.*

We show Theorem 3.1 by constructing a parametrix and refer the details of the proof to Ōuchi [7]. It follows from Theorem 3.1 that singularities of homogeneous solutions of $L(z, \partial_z)$ are of fractional order, provided they are in $\mathcal{O}_{\{\gamma\}}(U(\theta))$. So we assume $u(z) \in \mathcal{O}_{temp,c}U(\theta)$ in the following of this paper.

In order to analyze singularities, we make use of the Mellin transform with respect to z_0

$$\hat{u}(\lambda, z') = \int_0^T t^{\lambda-1}u(t, z')dt, \tag{3.2}$$

where T is a small positive constant. The transform (3.2) is Mellin transform on $arg z_0 = 0$, however, the Mellin transform on $\arg z_0 = \theta$ is also available to get the main result.

By the assumption $\hat{u}(\lambda, z')$ is holomorphic in $\{\lambda; \Re\lambda > -c\}$. It is the first aim to show it is meromorphically extensible to a larger region. Put $\Phi(\lambda, z') := \chi(-\lambda, z')$. We have

Theorem 3.2. *$\hat{u}(\lambda, z')$ $(z' \in V')$ is meromorphically extensible in λ to the whole λ-plain. Its poles are contained in $\cup_{n=0}^{\infty}\{\lambda; \Phi(z', \lambda + n) = 0\}$.*

Outline of the proof. $u(z)$ satisfies $A(z, \partial_{z_0})u(z) + B(z, \partial_z)u(z) = 0$, from which we have a partial differential difference equation $\hat{u}(\lambda, z')$ satisfies, that is, for any $N \in \mathbb{N}$

$$\begin{aligned}\Phi(\lambda, z')\hat{u}(\lambda, z') + \sum_{h=1}^{N} \mathcal{L}_h(\lambda, z', \partial')\hat{u}(\lambda + h, z')& \\ + \hat{u}_N(\lambda, z') + T^{\lambda}H_N(\lambda, z') = 0&,\end{aligned} \tag{3.3}$$

where $\mathcal{L}_h(\lambda, z', \partial')$ is a partial differential operator, whose coefficients are polynomial in λ. $H_N(\lambda, z')$ is a polynomial of λ and $\hat{u}_N(\lambda, z')$ is holomorphic in $\{\lambda; \Re\lambda > -N - c\}$. Equation (3.3) is obtained by the Mellin transform of the equation, integrations by parts and Taylor expansion of the coefficients. The order of Taylor expansion of the coefficients depends on N. We have easily the meromorphic extension by the relation (3.3).

Let us calculate the inverse Mellin transform and reconstruct $u(z)$. So the second aim is to obtain estimates of $\hat{u}(\lambda, z')$ outside of poles. Set

$$
\begin{aligned}
Z(r) &= \bigcup_{|z'|\le r} \bigcup_{n=0}^{\infty} \{\Phi(\lambda+n, z') = 0\} \\
Z(r,\delta) &= \{\lambda;\ d(\lambda, Z(r)) \le \varepsilon_0\},
\end{aligned} \tag{3.4}
$$

where $d(\lambda, A)$ means the distance of λ and set A. We choose $r > 0$ and $\delta > 0$ so small, if necessary. For $N \in \mathbb{N}$ set

$$\Lambda(N) = \{\lambda \notin Z(r', \varepsilon_0); -N + 1/2 - c \le \Re\lambda \le -N + 3/2 - c\}. \tag{3.5}$$

We have an estimate of $\hat{u}(\lambda, z')$ in $\Lambda(N)$

Proposition 3.3. *There are constants A, B and a polydisk V' such that for $z' \in V'$ and $\lambda \in \Lambda(N)$*

$$|\hat{u}(\lambda, z')| \le AB^N T^{\Re\lambda} \frac{\prod_{s=1}^{N}(|\lambda + N| + s)^m}{N!^m} \Gamma(\frac{N}{\gamma} + 1). \tag{3.6}$$

Let $\{\sigma_N\}_{N\in\mathbb{N}}$ be a sequence of real numbers such that the vertical line $\Re\lambda = -\sigma_N$ lies in $\Lambda(N)$. Define

$$u_N(z) = \frac{1}{2\pi i} \int_{C_N} z_0^{-\lambda} \hat{u}(\lambda, z') d\lambda, \tag{3.7}$$

where C_N is a contour which encloses all the poles of $\hat{u}(\lambda, z')$ in $\Re\lambda > -\sigma_N$. $u_N(z)$ gives asymptotic behavior of $u(z)$. We have

Theorem 3.4. *Let $u(z) \in \mathcal{O}_{temp}(U(\theta))$ be a solution of $L(z, \partial_z)u(z) = 0$ and $u_N(z)$ be the function defined by (3.7). Then there is a polydisk V centered at $z = 0$ such that for any θ' with $0 < \theta' < \theta$ and any $N \in \mathbb{N}$*

$$|u(z) - u_N(z)| \le AB^N |z_0|^{\sigma_N} \Gamma(\frac{N}{\gamma} + 1) \quad \text{in } V(\theta') \tag{3.8}$$

holds for some constants A and B depending on θ'.

To show the remainder estimate (3.8) consider

$$u_N^R(t, z') = \frac{1}{2\pi i} \int_{\Re\lambda = -\sigma_N} t^{-\lambda} \hat{u}(\lambda, z') d\lambda \quad \text{for } t > 0. \tag{3.9}$$

Then formally $u_N^R(t, z') = u(t, z') - u_N(t, z')$ holds for $t > 0$. However the convergence of the integral of (3.9) is vague, because the estimate of $\hat{u}(\lambda, z')$

in $\Lambda(N)$ obtained in Proposition 3.3 is of polynomial growth in $\Im\lambda$. So we do not calculate directly it. However by the assumption that $u(z)$ is holomorphic on the sectorial region $U(\theta)$ we can modify (3.9), estimate the difference $u(t,z') - u_N(t,z')$ by another method and get Theorem 3.4.

If $L(z,\partial_z)$ is an operator of Fuchsian type (see example 2), then $\gamma = \infty$, so it follows from (3.8) that $u(z) = \lim_{N\to\infty} u_N(z)$ in $V(\theta')$ for small z, which is a generalization of the result of Mandai and Tahara concerning the structure of homogeneous solutions of operator of Fuchsian type.

Corollary 3.5. *Let $u(z) \in \mathcal{O}_{temp}(U(\theta))$ be a solution of $L(z,\partial_z)u(z) = 0$ satisfying $|u(z)| \le A|z_0|^a$ in $U(\theta)$ for some $a > a_1$, a_1 being the constant in* (3.1). *Then there is a polydisk V centered at $z = 0$ such that for any θ' with $0 < \theta' < \theta$*

$$|u(z)| \le C\exp(-c|z_0|^{-\gamma}) \quad \text{in } V(\theta') \tag{3.10}$$

holds for some positive constants C and c.

We have Corollary 3.5 by showing that $\hat{u}(\lambda, z')$ has no poles.

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